

# Edge Excitations and Non-Abelian Statistics in the Moore-Read State: A Numerical Study in the Presence of Coulomb Interaction and Edge Confinement

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We study the ground state and low-energy excitations of fractional quantum Hall systems on a disk at filling fraction  $\nu = 5/2$ , with Coulomb interaction and background confining potential. We find the Moore-Read ground state is stable within a finite but narrow window in parameter space. The corresponding low-energy excitations contain a fermionic branch and a bosonic branch, with widely different velocities. A short-range repulsive potential can stabilize a charge  $+e/4$  quasi-hole at the center, leading to a different edge excitation spectrum due to the change of boundary conditions for Majorana fermions, clearly indicating the non-Abelian nature of the quasi-hole.

Fractional quantum Hall (FQH) liquids represent novel states of matter with non-trivial topological order [1], whose consequences include chiral edge excitations and fractionally charged bulk quasiparticles that obey Abelian or non-Abelian fractional statistics. It has been proposed that the non-Abelian quasiparticles can be used for quantum information storage and processing in an intrinsically fault-tolerant fashion [2, 3], in which information is stored by the degenerate ground states in the presence of these non-Abelian quasiparticles, and unitary transformations in this Hilbert space can be performed by braiding the quasiparticles [4, 5]. While many Abelian FQH states have been observed and studied in detail [1], thus far there have been relatively few candidates for the non-Abelian ones. The most promising candidate is the FQH state at Landau level filling fraction  $\nu = 5/2$  [6]. The leading candidate for the ground state of this system is the Moore-Read (MR) paired state [7], which has been shown [7, 8] to support fractionally charged, non-Abelian quasiparticles. The MR state received strong support from numerical studies using sphere or torus geometries [9]. It has been proposed that the non-Abelian nature of the quasiparticles in the MR state may be detected through interference experiments in edge transport [10, 11]. In order to have a quantitative understanding of the edge physics however, one needs to study the interplay between electron-electron interaction and confining potential, which may lead to edge structures that are more complicated than those predicted by the simplest theory [12]. This was found to be the case rather generically in the FQH regime [13].

In anticipation of experimental studies, in this work we perform detailed numerical studies of edge excitations in the  $5/2$  FQH state in finite-size systems with disc geometry, taking into account the inter-electron Coulomb interaction and a semi-realistic model of the confining potential due to neutralizing background charge. For a limited parameter space, we find the ground state has substantial overlap with the MR state. Within this parameter space we identify the existence of chiral fermionic and bosonic

edge modes, in agreement with previous prediction. We find the fermionic mode velocity is much lower than that of the bosonic mode. With suitable short-range repulsive potential at the center, we show that a charge  $+e/4$  quasi-hole can be localized at the center of the system, and its presence changes the spectrum of the fermionic edge mode. This confirms the existence and non-Abelian nature of such fractionally charged quasiparticles.

*The microscopic model.* We consider a microscopic model of a two-dimensional electron gas (2DEG) confined to a two-dimensional disk, with neutralizing background charge distributed uniformly on a parallel disk of radius  $a$  at a distance  $d$  above the 2DEG. This distance parameterizes the strength of the confining potential, which decreases with increasing  $d$ . For  $\nu = 5/2$ , we explicitly keep the electronic states in the first Landau level (1LL) only, while neglecting the spin up and down electrons in the lowest Landau level (0LL), assuming they are inert. The amount of positive background charge is chosen to be equal to that of the half-filled 1LL, so the system is neutral. The choice of  $a = \sqrt{4N}$ , in units of  $l_B$  (magnetic length), guarantees that the disk encloses exactly  $2N$  magnetic flux quanta for  $N = 2P$ , corresponding to  $\nu = 1/2$  in the 1LL [14]. The rotationally invariant confining potential comes from the Coulomb attraction between the background charge and the electrons. Using the symmetric gauge, we can write down the following Hamiltonian for the electrons confined to the 1LL:

$$H_C = \frac{1}{2} \sum_{mnl} V_{mn}^l c_{m+l}^\dagger c_n^\dagger c_{n+l} c_m + \sum_m U_m c_m^\dagger c_m, \quad (1)$$

where  $c_m^\dagger$  is the electron creation operator for the 1LL single electron state with angular momentum  $m$ ,  $V_{mn}^l$ 's are the corresponding matrix elements of Coulomb interaction for the symmetric gauge, and  $U_m$ 's are the matrix elements of the confining potential.

*MR ground state.* We diagonalize the Hamiltonian [Eq. (1)] for each Hilbert subspace with total angular momentum  $M$ , and obtain the ground state energy  $E(M)$ . Figure 1(a) shows  $E(M)$  vs  $M$  for  $N = 12$  electrons

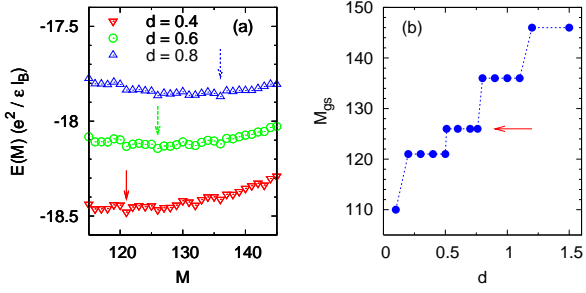


FIG. 1: (Color online) (a) Ground state energy  $E(M)$  in each angular momentum  $M$  subspace for  $N = 12$  electrons with 22 orbitals in the 1LL (corresponding to  $\nu = 1/2$ ). The positive background charge is at a distance  $d = 0.4, 0.6$ , and  $0.8$ , in units of  $l_B$ , above the electron plane. The total angular momentum of the global ground state  $M_{gs}$  (indicated by arrows) increases from 121, 126, to 136, respectively. The global ground state at  $d = 0.6$  has the same total angular momentum  $M_{MR} = N(2N - 3)/2 = 126$  as the corresponding  $N = 12$  MR state. The overlap between the two states is 0.47. The curves for  $d = 0.4$  and  $0.8$  have been shifted vertically by  $0.6$  and  $-0.6$ , respectively. (b) The total angular momentum of the global ground state  $M_{gs}$  as a function of  $d$ . The arrow indicates the plateau at which  $M_{gs} = M_{MR}$ .

in the 1LL in 22 orbitals, which is the minimum to accommodate the corresponding MR state. As illustrated in Fig. 1(b),  $M_{gs}$  increases with increasing  $d$ . This is very similar to what happens in the 0LL [13], and reflects the interplay between electron-electron Coulomb repulsion and confining potential; as the confining potential weakens with increasing  $d$ , electrons tend to move outward, resulting in bigger  $M_{gs}$ .  $M_{gs}$  coincides with the total angular momentum of the  $N = 12$  MR state  $M_{MR} = N(2N - 3)/2 = 126$  only within a small window:  $0.51 \leq d/l_B \leq 0.76$ . This contrasts with the situation for a Laughlin filling fraction  $\nu = 1/3$ , where  $M_{gs} = N(N - 1)/2\nu$  (same as the corresponding Laughlin state) for a substantially bigger window  $d < d_c \approx 1.5l_B$  [13]. When the global ground state has the same total angular momentum as the MR state, the overlap  $|\langle \Psi_{gs} | \Psi_{MR} \rangle|^2$  between the two is about 0.47. We note that, in the absence of confining potential, the corresponding overlaps are 0.46 and 0.45 for  $N = 12$  and  $14$ , respectively. These values are quite substantial, given that the size of Hilbert subspaces are 16,660 and 194,668. But they are well below the overlap ( $> 0.95$ ) in the case of the Laughlin filling  $\nu = 1/3$  [13]. The reduced overlap and window for the MR state reflect the fact that the paired state is much weaker compared to the Laughlin state. The overlap is, however, quite sensitive to small changes of system parameters; for example, we can increase the overlap to above 0.7 for  $N = 12$ , by choosing  $a = \sqrt{4N} - 4$ , and a change in the  $V_1$  pseudopotential,  $\delta V_1 = 0.03$ . Such sensitivity suggests that the MR state is rather “fragile”, consistent with experiments at  $\nu = 5/2$ .

*Edge excitations.* The MR state has non-trivial topological order. While our numerical results indicate that the ground state has substantial overlap with the MR state for properly chosen system parameters, it does not directly reflect the topological order of the system. One way to probe the topological order is to study edge excitations, which is also of vital experimental importance. For comparison, the Laughlin state supports one bosonic branch of chiral edge excitations, whose properties have been studied in tunneling experiments [15]. For  $\nu = 5/2$ , a neutral fermionic branch of excitations has been predicted in addition to a bosonic branch [1, 16]. The existence of both branches makes the low-energy excitation spectrum of a microscopic model at  $\nu = 5/2$  richer, and their experimental consequences more interesting [17].

Figure 2(a) shows the low-energy excitations for pure Coulomb interaction and the confining potential with  $d = 0.6$  for 12 electrons in 26 orbitals. Apparently, there is no clear distinction between fermionic and bosonic edge modes as well as bulk modes, due to the relatively small bulk gap, and system size. The situation here is similar to a related study on a rotating Bose gas [18], and will be analyzed in detail elsewhere. Here we focus instead on a model with mixed Coulomb interaction and 3-body interaction for clarity. The 3-body interaction alone generates the MR state as its exact ground state with the smallest total angular momentum. The mixed Hamiltonian is

$$H = (1 - \lambda)H_C + \lambda H_{3B}, \quad (2)$$

$$H_{3B} = - \sum_{i < j < k} S_{ijk} [\nabla_i^2 \nabla_j^4 \delta(\mathbf{r}_i - \mathbf{r}_j) \delta(\mathbf{r}_i - \mathbf{r}_k)], \quad (3)$$

where  $S$  is a symmetrizer:  $S_{123}[f_{123}] = f_{123} + f_{231} + f_{312}$ . We measure energies in units of  $e^2/\epsilon l_B$ . As we will see, the mixed interaction, which enhances the bulk excitation gap with respect to that of edge excitations, allows for a clear separation between the two in a finite system, effectively increasing the system size.

Figure 2(b) shows the low-energy excitations  $\Delta E(\Delta M)$  for 12 electrons in 26 orbitals in the 1LL for the mixed Hamiltonian with  $\lambda = 0.5$  and  $d = 0.6l_B$ . There is a clear separation of the spectrum around  $\Delta E = 0.1$ , below which we identify as edge modes. The total numbers of these states are 1, 1, 3, 5, and 10 for  $\Delta M = 0-4$ , which agree with the numbers of edge states expected for the MR state [16]. Notably, the lowest two levels for  $\Delta M = 4$  lie very close to each other.

In this case, we can further separate the fermionic and bosonic branches of the edge states. The procedure is similar to but more complicated than the one we used [13] to identify edge modes in the Laughlin case at  $\nu = 1/3$ , where there is only one bosonic branch of edge modes. The basic idea is to label the low-lying states by two sets of occupation numbers  $\{n_b(l_b)\}$  and  $\{n_f(l_f)\}$  for bosonic and fermionic modes with angular momentum  $l_{b,f}$ , respectively. Since the fermionic edge

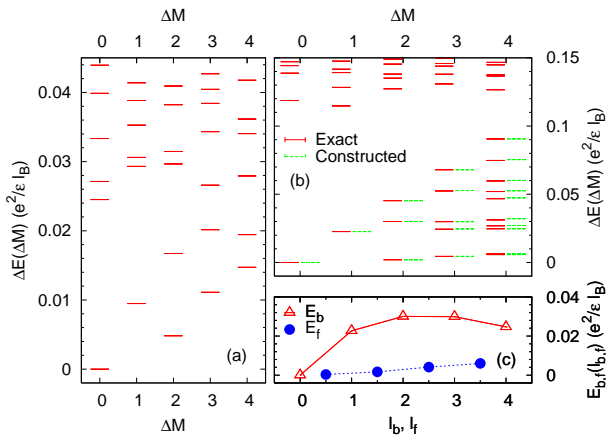


FIG. 2: (Color online) Low-energy excitations  $\Delta E(\Delta M)$  from exact diagonalization (solid lines) for  $N = 12$  electrons in 26 orbitals in the 1LL (corresponding to  $\nu = 1/2$ ) for (a) Coulomb Hamiltonian [Eq. (1)] and (b) mixed Hamiltonian [Eq. (2) with  $\lambda = 0.5$ ]. The neutralizing background charge for the Coulomb part is deposited at  $d = 0.6l_B$  above the electron plane. (c) Dispersion curves of bosonic ( $E_b$ ) and fermionic modes ( $E_f$ ) of the system. These energies can be used to construct the complete edge spectrum for the 12-electron system up to  $\Delta M = 4$  [dashed bars in (b)].

excitations are Majorana fermions that obey antiperiodic boundary conditions [16],  $l_f$  must be positive half integers and  $\sum_{l_f} n_f(l_f)$  an even integer because fermion modes are occupied in pairs. The angular momentum and energy of the state, measured from those of the ground state, are  $\Delta M = \sum_{l_b} n_b(l_b)l_b + \sum_{l_f} n_f(l_f)l_f$  and  $\Delta E = \sum_{l_b} n_b(l_b)E_b(l_b) + \sum_{l_f} n_f(l_f)E_f(l_f)$ . The difficulty here is, in addition to the bosonic modes, we also have fermionic modes, and thus the convolution of fermionic and bosonic modes. Fortunately, we note that the number of states with nearly zero energy coincides with the number of fermionic edge states expected by theory. Accordingly in our construction we will assume these energies to have been evolved from combining two Majorana fermions. Through careful analysis of the low-energy excitations up to  $\Delta M = 4$ , we obtain the results of bosonic and fermionic mode energies up to  $l_b = 4$  and  $l_f = 7/2$ , respectively, plotted in Fig. 2(c). The detailed analysis will be published elsewhere. Using these 8 energies [excluding the trivial  $E_b(0) = 0$ ], we can construct the whole low-energy spectrum of the system up to  $\Delta M = 4$ , a total of 20 states. The excellent agreement [see Fig. 2(b)] justifies our analysis, and thus our result, that, energetically, fermionic modes are well separated from bosonic modes. In contrast to the roughly linear dispersion of the fermionic branch, the energy of the bosonic branch bends down (despite a much bigger initial slope or higher velocity), suggesting a potential vulnerability to edge reconstruction in the bosonic branch [13]. These are not surprising since the bosonic modes are charged; as a result its velocity is dominated by the long-range nature of the

Coulomb interaction in the long-wavelength limit, but in the meantime it is also more sensitive to the competition between Coulomb interaction and confining potential which can lead to instability at shorter wavelength.

*Charge  $+e/4$  and  $+e/2$  quasiholes.* One of the most important properties of the MR state is that it supports charge  $\pm e/4$  quasihole/particle excitations. To demonstrate the unusual fractional charge, we add, to the mixed Hamiltonian with  $\lambda = 0.5$  and  $d = 0.5l_B$ , a short-range potential:  $H_W = Wc_0^\dagger c_0$ , which tends to create quasiparticles or quasiholes at the origin. For small enough repulsive  $W$ , the ground state of the system should remain MR-like. As  $W$  is increased, a quasihole of charge  $+e/4$  can appear at the origin, reflected by a change of ground state angular momentum from  $M_{gs} = N(2N - 3)/2$  to  $N(2N - 3)/2 + N/2$ , and depletion of  $1/4$  in the total occupation number of electrons at orbitals with small angular momenta. If  $W$  is increased further, a  $+e/2$  quasihole, much like a quasihole for the Laughlin state, appears near the origin in the global ground state, whose total angular momentum further increases to  $N(2N - 3)/2 + N$ . This is observed for a system of 12 electrons in 24 orbitals (as well as a smaller system of 10 electrons in 20 orbitals). Figure 3(a) shows the increase of  $M_{gs}$  from 126 to 132 and then to 138 with increasing  $W$ . Fig. 3(b) compares the electron occupation number  $n(m)$  in each orbital for  $W = 0.0$  ( $M_{gs} = 126$ , MR-like) and  $W = 0.1$  ( $M_{gs} = 132$ ). The accumulated difference in the occupation numbers of the two states,  $\sum_{i=0}^m \Delta n(i)$ , oscillates around  $-0.25$  for  $m$  up to about 19, indicating the existence of a  $+e/4$  quasihole at the origin. The same comparison for  $W = 0.1$  ( $M_{gs} = 132$ ) and  $W = 0.25$  ( $M_{gs} = 138$ ) is plotted in Fig. 3(c). Their difference ( $\sim -0.25$ ) indicates the emergence of another  $+e/4$  quasihole at the origin, or a  $+e/2$  quasihole compared to the MR-like state for  $W = 0.0$ .

The  $+e/4$  quasihole supports a zero energy Majorana fermion mode which is responsible for its non-Abelian nature. This zero mode pairs with the edge excitations and changes their spectra. With quasiholes in the bulk, the fermionic edge excitations are Majorana fermions that obey either periodic (integer  $l_f$ , twisted sector) or antiperiodic (half integer  $l_f$ , untwisted sector) boundary conditions [16]. In the presence of an odd number of  $+e/4$  quasiholes (twisted sector), two-fermion edge excitations are shifted by  $\delta(\Delta M) = -1$ , relative to those in the presence of an even number of such quasiholes (untwisted sector). This is demonstrated in Fig. 3(d)-(f) for 12 electrons in 24 orbitals. From  $M_{gs} = 126$  in Fig. 3(d) ( $W = 0.0$ , no quasihole present), we count the numbers of fermionic edge states as 0, 1, 1, 2 for  $\Delta M = 1-4$ . From  $M_{gs} = 132$  in Fig. 3(e) ( $W = 0.1$ , one  $+e/4$  quasihole present), the numbers change to 1, 1, 2, 2 for  $\Delta M = 1-4$ . In particular, the existence of a fermionic state at  $\Delta M = 1$  clearly indicates the change. From  $M_{gs} = 138$  in Fig. 3(f) ( $W = 0.25$ , two  $+e/4$  quasiholes present),

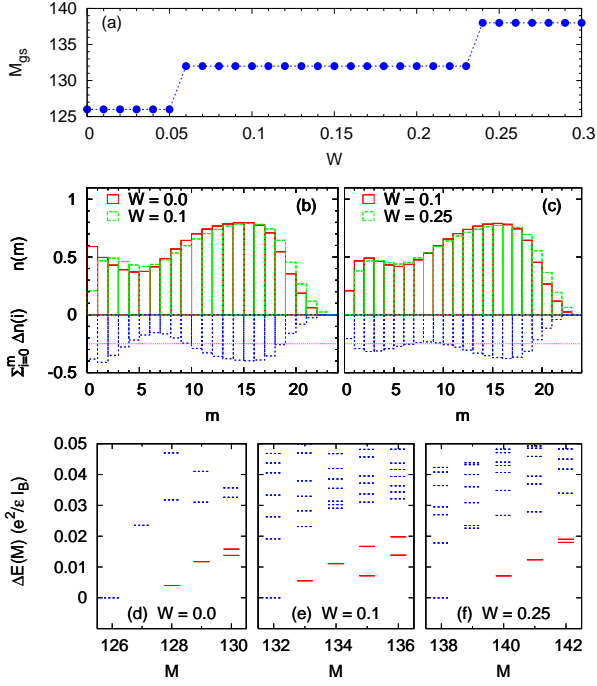


FIG. 3: (Color online) Generation of quasiholes using a short-range repulsion  $Wc_0^\dagger c_0$  in a system of 12 electrons in 24 orbitals, for the mixed Hamiltonian Eq. (2) with  $\lambda = 0.5$  and  $d = 0.5l_B$ . (a) Ground state angular momentum  $M_{gs}$  as a function of  $W$ . (b) Electron occupation number  $n(m)$  of the ground state for  $W = 0.0$  ( $M_{gs} = 126$ ) and  $W = 0.1$  ( $M_{gs} = 132$ ), as well as the accumulated difference in  $n(m)$  between the two states,  $\sum_{i=0}^m \Delta n(i)$ , which oscillates around  $-0.25$  (dotted line) for  $m$  up to about 19, indicating the emergence of a charge  $+e/4$  quasihole at  $m = 0$ . (c)  $n(m)$  for  $W = 0.1$  and  $W = 0.25$  ( $M_{gs} = 138$ ), and their accumulated difference, indicating the appearance of another  $+e/4$  quasihole. Low-energy edge excitations of the systems are plotted for (d)  $W = 0.0$ , (e)  $W = 0.1$ , and (f)  $W = 0.25$ . The fermionic mode supports 0, 1, 1, 2 states (solid bars) for  $\Delta M = 1-4$  in (d) ( $M_{gs} = 126$ ), but 1, 1, 2, 2 for  $\Delta M = 1-4$  in (e) ( $M_{gs} = 132$ ). The numbers change back to 0, 1, 1, 2 for  $\Delta M = 1-4$  in (f) ( $M_{gs} = 138$ ). This suggests that a single  $+e/4$  quasihole (or an odd number of quasiholes) changes the fermionic mode spectrum while a  $+e/2$  quasihole (or, in general, an even number of  $+e/4$  quasiholes) does not.

the numbers change back to 0, 1, 1, 2 for  $\Delta M = 1-4$ .

**Summary.** Our results suggest that the Moore-Read (MR) state properly describes a half-filled first Landau level, for properly chosen confinement potential. In this case the system supports chiral edge excitations as well as fractionally charged quasiholes, and their properties

agree with theory predictions. We also find that the window of stability of the MR state is rather narrow, and the edge modes may suffer from reconstruction or other instabilities as the confinement potential varies. The nature and consequences of such instabilities are currently under investigation, which will be presented elsewhere along with further details of the present work.

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  - [14] This is a simplification from the real system. In reality the background charge equals the *total* electron charge that includes the filled 0LL electrons; the latter are also separated from the background charge, and the location of their edge is different from that of the 1LL electrons. The consequences of these will be discussed elsewhere.
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